PROBABILITY

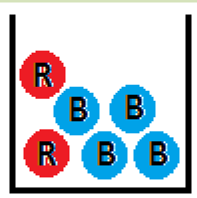
(in)dependent events

DEFINITION

Two events are **independent** if they have no influence on each other. In this case, the probability that they both occur is equal to the product of the probabilities of the two individual events: $P(A \cap B) = P(A) \times P(B)$.

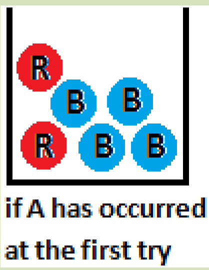
EXAMPLES

Consider a bag with colored marbles: 2 red and 4 blue. Two marbles are drawn from it consecutively.



A: "The first marble drawn is red."
B: "The second marble drawn is blue."
What is the probability of getting a red marble first, and then a blue one? (In other words, calculate $P(A \cap B)$.)

<u>1st case</u> <u>If the first marble is drawn with replacement</u>
Since the first marble is replaced, then events A and B are independent. The bag contains the same number of marbles before and after the first try.

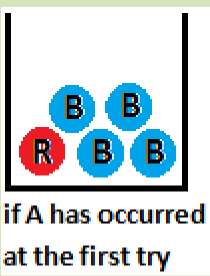


The chance of getting a blue marble during the second try is 4 in 6.
Then $P(A \cap B) = P(A) \times P(B)$

$$= \frac{2}{6} \times \frac{4}{6}$$

$$\boxed{P(A \cap B) = \frac{8}{36}}$$

<u>2nd case</u> <u>If the first marble is drawn without replacement</u>
In this case, the contents of the bag differ; events A and B are therefore dependent.



The chance of getting a blue marble during the second try is 4 in 5.
$P(A \cap B) = P(A) \times P(B|A)$ (multiplication rule)

$$= \frac{2}{6} \times \frac{4}{5}$$

$$\boxed{P(A \cap B) = \frac{8}{30}}$$

PHYSICAL INDEPENDENCE IMPLIES MATHEMATICAL INDEPENDENCE ... but the converse is not true

Events that are part of the same random procedure and not obviously physically independent may turn out to obey the defining relationship for mathematical independence, as the following example demonstrates.

Suppose a fair die is rolled twice. Consider the following two events:
A= "a three is obtained on the second roll"
B= "the sum of the two numbers obtained is less or equal to four"

In this case, somewhat surprisingly, we can show that $P(B|A) = P(B)$. Thus A and B are mathematically independent, even though at face value they seem related.

To see this, we write the elementary events in this random process as (x, y), with x the result of the first roll and y the result of the second roll. Then:
$A = \{(1,3), (2,3), (3,3), (4,3), (5,3), (6,3)\}$
$B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

There are 36 elementary events, each with probability $\frac{1}{36}$.
So $P(A) = \frac{6}{36} = \frac{1}{6}$ and $P(B) = \frac{6}{36} = \frac{1}{6}$.
We also have $A \cap B = \{(1,3)\}$ and so $P(A \cap B) = \frac{1}{36}$.
Hence $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} = P(B)$.

Thus A and B are independent. Alternatively, we can check independence by calculating
$P(A \cap B) = \frac{1}{36} = \frac{1}{6} \times \frac{1}{6} = P(A) \times P(B)$.

source: AMSI

EXERCISE

For each of the following situations, discuss whether or not the event A and B should be regarded as independent:

A: "the tenth coin toss in a sequence of tosses of a fair coin is a head"
B: "the first nine tosses in the sequence each result in a head"

A: "the weather is rainy"
B: "I take my umbrella"

A: "the weather is rainy"
B: "there is an episode of *Little house on the prairie* on TV"