Table II. Relation of Operating Characteristics and Equations

Symbol	In Terms of Operation	In Terms of Nonoperation
X	The switch or relay X is operated	The switch or relay X is not operated
$=$	If	If
X'	The switch or relay X is not operated	The switch or relay X is operated
$\cdot$	Or	And
$+$	And	Or
$(--)'$	The circuit $(--)$ is not closed, or apply De Morgan's theorem	The circuit $(--)$ is closed, or apply De Morgan's theorem
$X(t-p)$	X has been operated for at least p seconds	X has been open for at least p seconds

If the dependent variable appears in its own defining function (as in a lock-in circuit) strict adherence to the above leads to confusing sentences. In such cases the following equivalents should be used.

$X = RX + S$	X is operated when R is closed (providing S is closed) and remains so independent of R until S opens	
$X = (R' + X)S'$		X is opened when R is closed (providing S is closed) and remains so independent of R until S opens

In using this table it is usually best to write the function under consideration either as a sum of pure products or as a product of pure sums. In the case of a sum of products the characteristics should be defined in terms of nonoperation; for a product of sums in terms of operation. If this is not done it is difficult to give implicit and explicit parentheses the proper significance.

conditions may be either conditions for operation or for nonoperation. Equations are written from operating characteristics according to Table II. To illustrate the use of this table suppose a relay U is to operate if x is operated and y or z is operated and v or w or z is not operated. The expression for A will be

$$U = x + yz + v'w'z' .$$

Lock-in relay equations have already been discussed. It does not, of course, matter if the same conditions are put in the expression more than once — all superfluous material will disappear in the final simplification.

3. The expressions for the various dependent variables should next be simplified as much as possible by means of the theorems on manipulation of these quantities. Just how much this can be done depends somewhat on the ingenuity of the designer.

4. The resulting circuit should now be drawn. Any necessary additions dictated by practical considerations such as current-carrying ability, sequence of contact operation, etc., should be made.

V. Illustrative Examples

In this section several problems will be solved with the methods which have been developed. The examples are intended more to illustrate the use of the calculus in actual problems and to show the versatility of relay and switching circuits than to describe practical devices.

It is possible to perform complex mathematical operations by means of relay circuits. Numbers may be represented by the positions of relays or stepping switches, and interconnections between sets of relays can be made to represent various mathematical operations. In fact, any operation that can be completely described in a finite number of steps using the words "if," "or," "and," etc. (see Table II), can be done automatically with relays. The last example is an illustration of a mathematical operation accomplished with relays.

A Selective Circuit

A relay U is to operate when any one, any three or when all four of the relays w, x, y and z are operated but not when none or two are operated. The hindrance function for U will evidently be:

$$U = wxyz + w'x'yz + w'xy'z + w'xyz' + wx'y'z + wx'yz' + wxy'z' \ .$$

Reducing to the simplest series-parallel form:

$$U = w[x(yz + y'z') + x'(y'z + yz')] + w'[x(y'z + yz') + x'yz] \ .$$

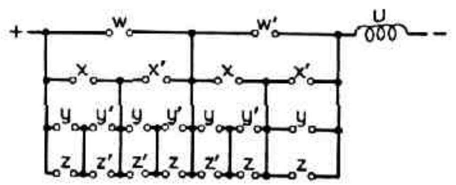

Figure 30. Series-parallel realization of selective circuit

This circuit is shown in Figure 30. It requires 20 elements. However, using the symmetric-function method, we may write for U:

$$U = S_{1,3,4}(w,x,y,z) \ .$$

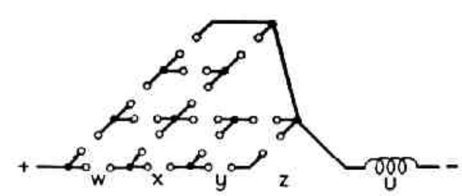

Figure 31. Selective circuit from symmetric-function method

This circuit (Figure 31) contains only 15 elements. A still further reduction may be made with the following device. First write

$$U' = S_{0,2}(w,x,y,z) \ .$$

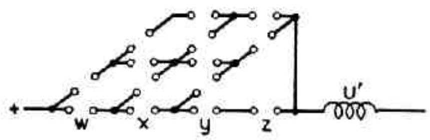

Figure 32. Negative of selective circuit from symmetric-function method

This has the circuit of Figure 32. What is required is the negative of this function. This is a planar network and we may apply the theorem on the dual of a network, thus obtaining the circuit shown in Figure 33. This contains 14 elements and is probably the most economical circuit of any sort.

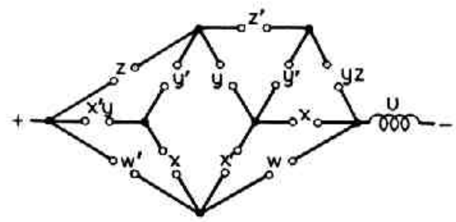

Figure 33. Dual of figure 32

Design of an Electric Combination Lock

An electric lock is to be constructed with the following characteristics. There are to be five pushbutton switches available on the front of the lock. These will be labeled a, b, c, d, e. To operate the lock the buttons must be pressed in the following order: c, b, a and c simultaneously, d. When operated in this sequence the lock is to unlock, but if any button is pressed incorrectly an alarm U is to operate. To relock the system a switch g must be operated. To release the alarm once it has started a switch h must be operated. This being a sequential system either a stepping switch or additional sequential relays are required. Using sequential relays let them be denoted by w, x, y and z corresponding respectively to the correct sequence of operating the push buttons. An additional time-delay relay is also required due to the third step in the operation. Obviously, even in correct operation a and c cannot be pressed at exactly the same time, but if only one is pressed and held down the alarm should operate. Therefore assume an auxiliary time delay relay v which will operate if either a or c alone is pressed at the end of step 2 and held down longer than time s, the delay of the relay.

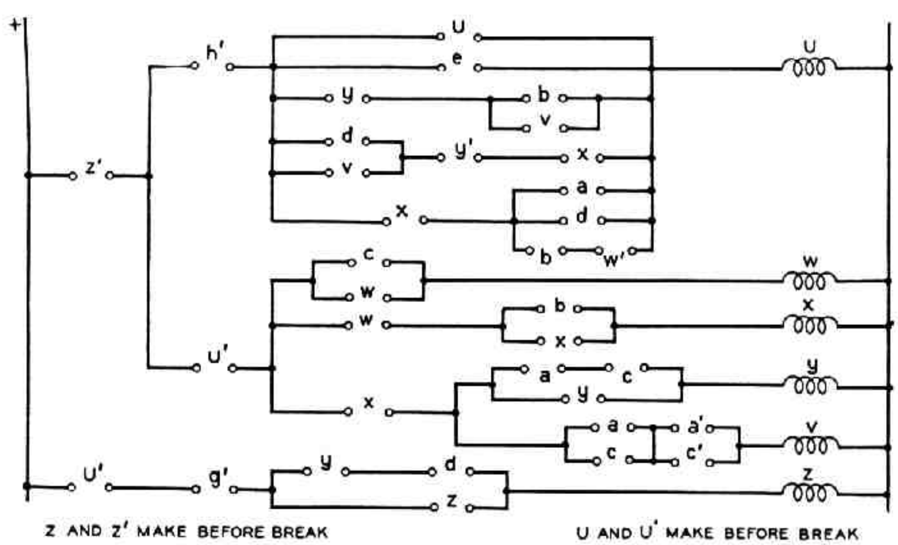

Figure 34. Combination-lock circuit

When z has operated the lock unlocks and at this point let all the other relays drop out of the circuit. The equations of the system may be written down immediately:

$$w = cw + z' + U' \ ,$$

$$x = bx + w + z' + U' \ ,$$

$$y = (a + c)y + x + z' + U' \ ,$$

$$z = z(d + y) + g' + U' \ ,$$

$$v = x + ac + a'c' + z' + U' \ ,$$

$$U = e(w' + abd)(w + x' + ad)[x + y' + dv(t - s)][y + bv(t - s)]U + h' + z' \ .$$

These expressions can be simplified considerably, first by combining the second and third factors in the first term of U, and then by factoring out the common terms of the several functions. The final simplified form is as below: This corresponds to the circuit of Figure 34.

$$\begin{array}{l} U = \\ w = \\ x = Z' + \\ y = \\ v = \end{array} \left| \; U' + \; \right| \begin{array}{l} h' + e[ad(b + w') + x'] \cdot \\ (x + y' + dv)(y + vb)U \\ \left| \begin{array}{l} cw \\ bx + w \\ x + \left| \begin{array}{l} (a + c)y \\ ac + a'c' \end{array} \right. \end{array} \right. \end{array}$$

$$z = g' + (y + d)z + U'$$

Electric Adder to the Base Two

A circuit is to be designed that will automatically add two numbers, using only relays and switches. Although any numbering base could be used the circuit is greatly simplified by using the scale of two. Each digit is thus either 0 or 1; the number whose digits in order are $a_k, a_{k-1}, a_{k-2}, \ldots a_2, a_1, a_0$ has the value $\sum_{j=0}^{k} a_j 2^j$.

Let the two numbers which are to be added be represented by a series of switches: $a_k, a_{k-1}, \ldots a_1, a_0$ representing the various digits of one of the numbers and $b_k, b_{k-1}, \ldots b_1, b_0$ the digits of the other number. The sum will be represented by the positions of a set of relays $s_{k+1}, s_k, s_{k-1} \ldots s_1, s_0$. A number which is carried to the jth column from the $(j-1)$th column will be represented by a relay c_j. If the value of any digit is zero, the corresponding relay or switch will be taken to be in the position of zero hindrance; if one, in the position where the hindrance is one. The actual addition is shown below:

c_{k+1}	$c_k \quad c_{j+1}c_j \quad c_2c_1$	Carried numbers
	$a_k\text{---}a_{j+1}a_j\text{---}a_2a_1a_0$	First number
	$b_k \quad b_{j+1}b_j \quad b_2b_1b_0$	Second number
c_{k+1} or s_{k+1}	$s_k\text{---}s_{j+1}s_j\text{---}s_2,s_1s_0$	Sum

Starting from the right, s_0 is one if a_0 is one and b_0 is zero or if a_0 is zero and b_0 one but not otherwise. Hence

$$s_0 = a_0b_0' + a_0'b_0 = a_0 \oplus b_0 .$$

c_1 is one if both a_0 and b_0 are one but not otherwise:

$$c_1 = a_0 \cdot b_0 .$$

s_j is one if just one of a_j, b_j, c_j is one, or if all three are one:

$$s_j = S_{1,3}(a_j, b_j, c_j) , \quad j = 1, 2, \ldots k .$$

c_{j+1} is one if two or if three of these variables are one:

$$c_{j+1} = S_{2,3}(a_j, b_j, c_j) , \quad j = 1, 2, \ldots k .$$

Using the method of symmetric functions, and shifting down for s_j gives the circuits of Figure 35. Eliminating superfluous elements we arrive at Figure 36.

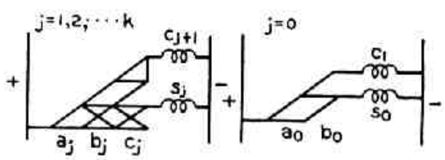

Figure 35. Circuits for electric adder

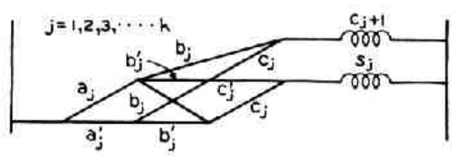

Figure 36. Simplification of figure 35

References

1. A complete bibliography of the literature of symbolic logic is given in the *Journal of Symbolic Logic*, volume 1, number 4, December 1936. Those elementary parts of the theory that are useful in connection with relay circuits are well treated in the two following references.

2. The Algebra of Logic, **Louis Cauturat.** The Open Court Publishing Company.

3. Universal Algebra, **A. N. Whitehead.** Cambridge, at the University Press, volume I, book III, chapters I and II, pages 35-42.

4. **E. V. Huntington,** *Transactions* of the American Mathematical Society, volume 35, 1933, pages 274-304. The postulates referred to are the fourth set, given on page 280.