all the relays $X_1,\ldots X_n$ the same. So far we have only considered transformations which may be applied to a two-terminal network keeping the operation of one relay in series with this network the same. To this end we define equivalence of n-terminal networks as follows. Definition: Two n-terminal networks M and N will be said to be equivalent with respect to these n terminals if and only if $X_{jk} = Y_{jk}$; $j,k = 1,2,3\ldots.n$, where X_{jk} is the hindrance of N (considered as a two-terminal network) between terminals j and k, and Y_{jk} is that for M between the corresponding terminals. Under this definition the equivalences of the preceding sections were with respect to two terminals.

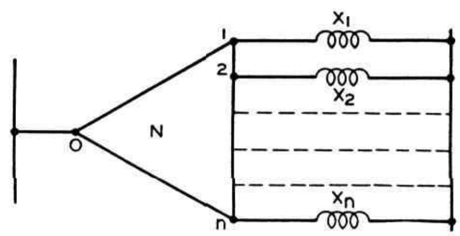

Figure 7. General constant-voltage relay circuit

Star-Mesh and Delta-Wye Transformations

As in ordinary network theory there exist star-to-mesh and delta-to-wye transformations. In impedance circuits these transformations, if they exist, are unique. In hindrance networks the transformations always exist and are not unique. Those given here are the simplest in that they require the least number of elements. The delta-to-wye transformation is shown in Figure 8. These two networks are equivalent with respect to the three terminals a,b, and c, since by distributive law $X_{ab} = R(S + T) = RS + RT$ and similarly for the other pairs of terminals $a-c$ and $b-c$.

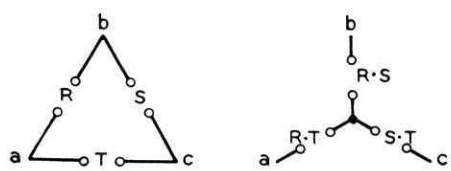

Figure 8. Delta-wye transformation

The wye-to-delta transformation is shown in Figure 9. This follows from the fact that $X_{ab} = R + S = (R + S)\cdot(R + T + T + S)$, etc. An n-point star also has a mesh equivalent with the central junction point eliminated. This is formed exactly as in the simple three-point star, by connecting each pair of terminals of the mesh through a hindrance which is the sum of the corresponding arms of the star. This may be proved by mathematical induction. We have shown it to be true for $n = 3$. Now assuming it true for $n - 1$, we shall prove it for n. Suppose we construct a mesh circuit from the given n-point star according to this method. Each corner of the mesh will be an $(n - 1)$-point star and since we have assumed the theorem true for $n - 1$ we may replace the nth corner by its mesh equivalent. If Y_{0j} was the hindrance of the original star from the central node 0 to the point j, then the reduced mesh will have the hindrance $(Y_{0s} + Y_{or})\cdot(Y_{0s} + Y_{on} + Y_{0r} + Y_{0n})$ connecting nodes r and s. But this reduces to $Y_{0s}Y_{0r}$ which is the correct value, since the original n-point star with the nth arm deleted becomes an $(n - 1)$-point star and by our assumption may be replaced by a mesh having this hindrance connecting nodes r and s. Therefore the two networks are equivalent with respect to the first $n - 1$ terminals. By eliminating other nodes than the nth, or by symmetry, the equivalence with respect to all n terminals is demonstrated.

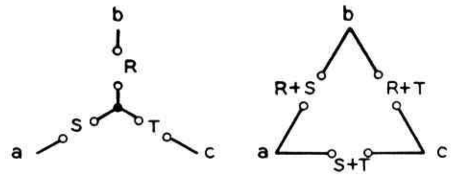

Figure 9. Wye-delta transformation

Hindrance Function of a Non-Series-Parallel Network

The methods of Part II were not sufficient to handle circuits which contained connections other than those of a series-parallel type. The "bridge" of Figure 10, for example, is a non-series-parallel network. These networks will be treated by first reducing to an equivalent series-parallel circuit. Three methods have been developed for finding the equivalent of a network such as the bridge.

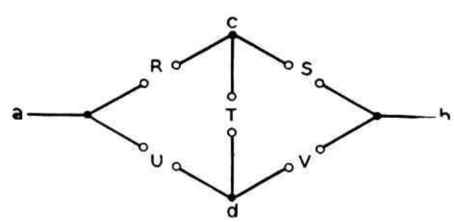

Figure 10. Non-series-parallel circuit

The first is the obvious method of applying the transformations until the network is of the series-parallel type and then writing the hindrance function by inspection. This process is exactly the same as is used in simplifying the complex impedance networks. To apply this to the circuit of Figure 10, first we may eliminate the node c, by applying the star-to-mesh transformation to the star $a-c$, $b-c$, $d-c$. This gives the network of Figure 11. The hindrance function may be written down from inspection for this network:

$$X_{ab} = (R + S)[U(R + T) + V(T + S)] .$$

This may be written as

$$X_{ab} = RU + SV + RTV + STU = R(U + TV) + S(V + TU) .$$

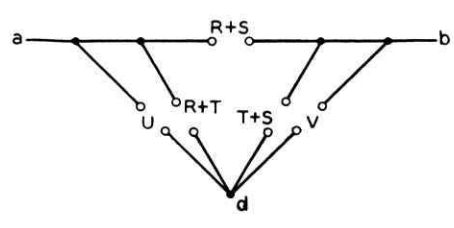

Figure 11. Hindrance function by means of transformations

The second method of analysis is to draw all possible paths through the network between the points under consideration. These paths are drawn along the lines representing the component hindrance elements of the circuit. If any one of these paths has zero hindrance, the required function must be zero. Hence if the result is written as a product, the hindrance of each path will be a factor of this product. The required result may therefore be written as the product of the hindrances of all possible paths between the two points. Paths which touch the same point more than once need not be considered. In Figure 12 this method is applied to the bridge. The paths are shown dotted. The function is therefore given by

$$X_{ab} = (R + S)(U + V)(R + T + V)(U + T + S)$$
$$= RU + SV + RTV + UTS = R(U + TV) + S(V + TU) .$$

The same result is thus obtained as with the first method.

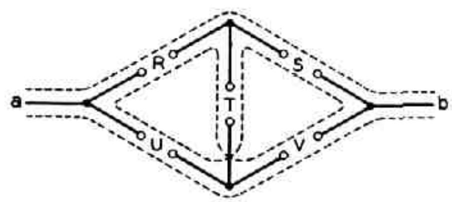

Figure 12. Hindrance function as a product of sums

The third method is to draw all possible lines which would break the circuit between the points under consideration, making the lines go through the hindrances of the circuit. The result is written as a sum, each term corresponding to a certain line. These terms are the products of all the hindrances on the line. The justification of the method is similar to that for the second method. This method is applied to the bridge in Figure 13.

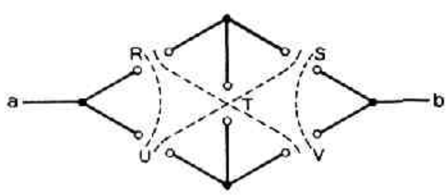

Figure 13. Hindrance function as a sum of products

This again gives for the hindrance of the network:

$$X_{ab} = RU + SV + RTV + STU = R(U + TV) + S(V + TU) .$$

The third method is usually the most convenient and rapid, for it gives the result directly as a sum. It seems much easier to handle sums than products due, no doubt, to the fact that in ordinary algebra we have the distributive law $X(Y + Z) = XY + XZ$, but not its dual $X + YZ = (X + Y)(X + Z)$. It is, however, sometimes difficult to apply the third method to nonplanar networks (networks which cannot be drawn on a plane without crossing lines) and in this case one of the other two methods may be used.

Simultaneous Equations

In analyzing a given circuit it is convenient to divide the various variables into two classes. Hindrance elements which are directly controlled by a source external to the circuit under consideration will be called independent variables. These will include hand-operated switches, contacts on external relays, etc. Relays and other devices controlled by the network will be called dependent variables. We shall, in general, use the earlier letters of the alphabet to represent independent variables and the later letters for dependent variables. In Figure 7 the dependent variables are $X_1, X_2 \ldots X_n$. X_k will evidently be operated if and only if $X_{0k} = 0$, where X_{0k} is the hindrance function of N between terminals 0 and k. That is,

$$X_k = X_{0k} \ , \quad k = 1,2,\ldots n \ .$$

This is a system of equations which completely define the operation of the system. The right-hand members will be known functions involving the various dependent and independent variables and given the starting conditions and the values of the independent variables the dependent variables may be computed.

A transformation will now be described for reducing the number of elements required to realize a set of simultaneous equations. This transformation keeps $X_{0k}(k = 1,2\ldots n)$ invariant, but $X_{jk}(j,k = 1,2\ldots n)$ may be changed, so that the new network may not be equivalent in the strict sense defined to the old one. The operation of all the relays will be the same, however, This simplification is only applicable if the X_{0k} functions are written as sums and certain terms are common to two or more equations. For example, suppose the set of equations is as follows:

$$W = A + B + CW \ ,$$

$$X = A + B + WX \ ,$$

$$Y = A + CY \ ,$$

$$Z = EZ + F \ .$$

This may be realized with the circuit of Figure 14, using only one A element for the three places where A occurs and only one B element for its two appearances. The justification is quite obvious. This may be indicated symbolically by drawing a vertical line after the terms common to the various equations, as shown below.

$$\begin{array}{ll|l|l} W = & & B + & CW \\ X = & A + & & WX \\ Y = & & CY & \\ Z = & F + EZ & & \end{array}$$

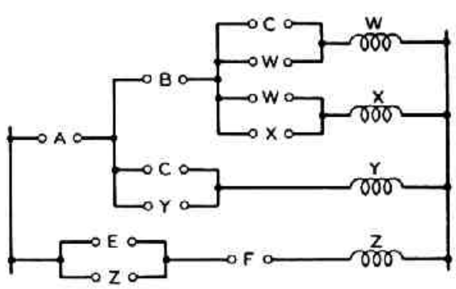

Figure 14. Example of reduction of simultaneous equations

It follows from the principle of duality that if we had defined multiplication to represent series connection, and addition for parallel connection, exactly the same theorems of manipulation would be obtained. There were two reasons for choosing the definitions given. First, as has been mentioned, it is easier to manipulate sums than products and the transformation just described can only be applied to sums (for constant-current relay circuits this condition is exactly reversed), and second, this choice makes the hindrance functions closely analogous to impedances. Under the alternative definitions they would be more similar to admittances, which are less commonly used.

Sometimes the relation $XY' = 0$ obtains between two relays X and Y. This is true if Y can operate only if X is operated. This frequently occurs in what is known as a sequential system. In a circuit of this type the relays can only operate in a certain order or sequence, the operation of one relay in general "preparing" the circuit so that the next in order can operate. If X precedes Y in the sequence and both are constrained to remain operated until the sequence is finished then this condition will be fulfilled. In such a case the following equations hold and may sometimes be used for simplification of expressions. If $XY' = 0$, then

$$X'Y' = Y' ,$$

$$XY = X ,$$

$$X' + Y = 1 ,$$

$$X' + Y' = X' ,$$

$$X + Y = Y .$$

These may be proved by adding $XY' = 0$ to the left-hand member or multiplying it by $X' + Y = 1$, thus not changing the value. For example, to prove the first one, add XY' to $X'Y'$ and factor.

Special Types of Relays and Switches

In certain types of circuits it is necessary to preserve a definite sequential relation in the operation of the contacts of a relay. This is done with make-before-break (or continuity) and break-make (or transfer) contacts. In handling this type of circuit the simplest method seems to be to assume in setting up the equations that the make and break contacts operate simultaneously, and after all simplifications of the equations have been made and the resulting circuit drawn, the required type of contact sequence is found from inspection.

Relays having a time delay in operating or deoperating may be treated similarly or by shifting the time axis. Thus if a relay coil is connected to a battery through a hindrance X, and the relay has a delay of p seconds in operating and releasing, then the hindrance function of the contacts of the relay will also be X, but at a time p seconds later. This may be indicated by writing $X(t)$ for the hindrance in series with the relay, and $X(t - p)$ for that of the relay contacts.

There are many special types of relays and switches for particular purposes, such as the stepping switches and selector switches of various sorts, multiwinding relays, cross-bar switches, etc. The operation of all these types may be described with the words "or," "and," "if," "operated," and "not operated." This is a sufficient condition that they may be described in terms of hindrance functions with the operations of addition, multiplication, negation, and equality. Thus a two-winding relay might be so constructed that it is operated if the first *or* the second winding is operated (activated) and the first *and* the second windings are not operated. If the first winding is X and the second Y, the hindrance function of make contacts on the relay will then be $XY + X'Y'$. Usually, however, these special relays occur only at the end of a complex circuit and may be omitted entirely from the calculations to be added after the rest of the circuit is designed.

Sometimes a relay X is to operate when a circuit R closes and to remain closed independent of R until a circuit S opens. Such a circuit is known as a lock-in circuit. Its equation is:

$$X = RX + S .$$

Replacing X by X' gives:

$$X' = RX' + S$$

or

$$X = (R' + X)S' .$$

In this case X is *opened* when R closes and remains open until S opens.

IV. Synthesis of Networks

Some General Theorems on Networks and Functions

It has been shown that any function may be expanded in a series consisting of a sum of products, each product being of the form $X_1 X_2 \ldots X_n$ with some permutation of primes on the letters, and each product having the coefficient 0 or 1. Now since each of the n variables may or may not have a prime, there is a total of 2^n different products of this form. Similarly each product may have the coefficient 0 or the coefficient 1 so there are 2^{2^n} possible sums of this sort. Hence we have the theorem: The number of functions obtainable from n variables is 2^{2^n}.

Each of these sums will represent a different function, but some of the functions may actually involve fewer than n variables (that is, they are of such a form that for one or more of the n variables, say X_k, we have identically $f|_{X_k=0} = f|_{X_k=1}$ so that under no conditions does the value of the function depend on the value X_k). Thus for two variables, X and Y, among the 16 functions obtained will be $X, Y, X', Y', 0$, and 1 which do not involve both X and Y. To find the number of functions which actually involve all of the n variables we proceed as follows. Let $\phi(n)$ be the number. Then by the theorem just given:

$$2^{2^n} = \sum_{k=0}^{n} \binom{n}{k} \phi(k) ,$$

where $\binom{n}{k} = n!/k!(n-k)!$ is the number of combinations of n things taken k at a time. That is, the total number of functions obtainable from n variables is equal to the sum of the numbers of those functions obtainable from each possible selection of variables from these n which actually involve all the variables in the selection. Solving for $\phi(n)$ gives

$$\phi(n) = 2^{2^n} - \sum_{k=0}^{n-1} \binom{n}{k} \phi(k) .$$

By substituting for $\phi(n-1)$ on the right the similar expression found by replacing n by $n-1$ in this equation, then similarly substituting for $\phi(n-2)$ in the expression thus obtained, etc., an equation may be obtained involving only $\phi(n)$. This equation may then be simplified to the form

$$\phi(n) = \sum_{k=0}^{n} \binom{n}{k} 2^{2^k} (-1)^{n-k} .$$

As n increases this expression approaches its leading term 2^{2^n} asymptotically. The error in using only this term for $n = 5$ is less than 0.01 percent.

We shall now determine those functions of n variables which require the most relay contacts to realize, and find the number of contacts required. In order to do this, it is necessary to define a function of two variables known as the sum modulo two or disjunct of the variables. This function is written $X_1 \oplus X_2$ and is defined by the equation:

$$X_1 \oplus X_2 = X_1 X_2' + X_1' X_2 .$$

It is easy to show that the sum modulo two obeys the commutative, associative, and the distributive law with respect to multiplication, that is,

$$X_1 \oplus X_2 = X_2 \oplus X_1 ,$$

$$(X_1 \oplus X_2) \oplus X_3 = X_1 \oplus (X_2 \oplus X_3) ,$$

$$X_1 (X_2 \oplus X_3) = X_1 X_2 \oplus X_1 X_3 .$$

Also

$$(X_1 \oplus X_2)' = X_1 \oplus X_2' = X_1' \oplus X_2 ,$$

$$X_1 \oplus 0 = X_1 ,$$

$$X_1 \oplus 1 = X_1' .$$

Since the sum modulo two obeys the associative law, we may omit parentheses in a sum of several terms without ambiguity. The sum modulo two of the n variables $X_1, X_2 \ldots X_n$ will for convenience be written:

$$X_1 \oplus X_2 \oplus X_3 \ldots \oplus X_n = \mathop{\Xi}_{k=1}^{n} X_k .$$

*Theorem:** The two functions of n variables which require the most elements (relay contacts) in a series-parallel realization are $\mathop{\Xi}_1^n X_k$ and $(\mathop{\Xi}_1^n X_k)'$, each of which requires $(3 \cdot 2^{n-1} - 2)$ elements.

This will be proved by mathematical induction. First note that it is true for $n = 2$. There are ten functions involving two variables, namely, XY, $X + Y$, $X'Y$, $X' + Y$, XY', $X + Y'$, $X'Y'$, $X' + Y'$, $XY' + X'Y$, $XY + X'Y'$. All of these but the last two require two elements; the last two require four elements and are $X \oplus Y$ and $(X \oplus Y)'$, respectively. Thus the theorem is true for $n = 2$. Now assuming it true for $n - 1$, we shall prove it true for n and thus complete the induction. Any function of n variables may be expanded about the nth variable as follows:

$$f(X_1, X_2 \ldots X_n) = f = X_n f(X_1 \ldots X_{n-1}, 1) + X_n' f(X_1 \ldots X_{n-1}, 0) . \qquad (19)$$

Now the terms $f(X_1 \ldots X_{n-1}, 1)$ and $f(X_1 \ldots X_{n-1}, 0)$ are functions of $n - 1$ variables and if they individually require the most elements for $n - 1$ variables, then f will require the most elements for n variables, providing there is no other method of writing f so that fewer elements are required. We have assumed that the most elements for $n - 1$ variables are required by $\mathop{\Xi}_1^{n-1} X_k$ and its negative. If we, therefore, substitute for $f(X_1 \ldots X_{n-1}, 1)$ the function $\mathop{\Xi}_1^{n-1} X_k$ and for $f(X_1 \ldots X_{n-1}, 0)$ the function $(\mathop{\Xi}_1^{n-1} X_k)'$ we find

* See the Notes to this paper.