The GOLDEN RATIO: PROPERTIES

ϕ has strange properties : multiplying it by itself, for instance, is exactly the same as adding one :$\phi^2 = \phi + 1$.

In other words, $\frac{1}{\phi} = \phi - 1$.

Now, let's try to find out a number A such that $A = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\ldots}}}}}$, in which the three little dots … mean that you go on forever.

Raise it to the square, you'll get $A^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\ldots}}}} = 1 + A$.

Therefore, $^2 = A + 1$: this is the first equation evoked above, whose solution is ϕ.

Likewise, it's easy to show that $\phi = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}$.

Last, it's impossible to write ϕ as a ratio of two whole numbers, so mathematicians call it irrational. This can be proved by contradiction.

In a proof by contradiction, we assume the logical negation of the result we wish to prove, and then reach some kind of contradiction. That means the assumption is false.

Assume that ϕ is a rational number, meaning that there exists 2 whole numbers a and b so that ϕ= $\frac{a}{b}$ and such that the fraction is shortened as much as possible. Therefore a and b can't be both even numbers.

From the properties of ϕ, we deduce $a^2 = b^2+ab$, thus $a^2-b^2=ab$.

If a and b are both odd numbers, we can't have $a^2-b^2=ab$. If a is even and b is odd or if a is odd and b is even either.

We infer ϕ is not a rational number.

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1. Prove that : $\phi^2 = \phi + 1$ then that $\frac{1}{\phi} = \phi - 1$.

2. Explain why $A^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\ldots}}}} = 1 + A$, then justify that A is ϕ.

3. Find three different writing for ϕ.

4. In the proof by contradiction of the irrationality of ϕ: what would happen if a and b were both even numbers?

5. Explain :

   a. "From the properties of ϕ, we deduce $a^2 = b^2+ab$, thus $a^2-b^2=ab$."

   b. "If a and b are both odd numbers, we can't have $a^2-b^2=ab$. If a is even and b is odd or if a is odd and b is even either."

6. Prove by contradiction that $\sqrt{2}$ is an irrational number as well.