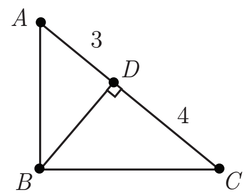
AMC 10 Student Practice Questions

You will find these and additional problems for the AMC 10 and AMC 12 on AMC's web site: http://www.unl.edu/amc, available from the current and previous AMC 10/12 Teacher Manuals, (http://www.unl.edu/amc/e-exams/e6-amc12/archive12.shtml) or from our Problems page archives (http://www.unl.edu/amc/a-activities/a7-problems/problem81012archive.shtml).

Triangle ABC has a right angle at B. Point D is the foot of the altitude from B, $AD = 3$, and $DC = 4$. What is the area of $\triangle ABC$?



(A) $4\sqrt{3}$ **(B)** $7\sqrt{3}$ **(C)** 21 **(D)** $14\sqrt{3}$ **(E)** 42

2009 AMC 10 A, Problem #10—
"$\triangle ADB$ and $\triangle BDC$ are similar."

Solution

Answer (B): By the Pythagorean Theorem, $AB^2 = BD^2 + 9$, $BC^2 = BD^2 + 16$, and $AB^2 + BC^2 = 49$. Adding the first two equations and substituting gives $2 \cdot BD^2 + 25 = 49$. Then $BD = 2\sqrt{3}$, and the area of $\triangle ABC$ is $\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}$.

OR

Because $\triangle ADB$ and $\triangle BDC$ are similar, $\frac{BD}{3} = \frac{4}{BD}$, from which $BD = 2\sqrt{3}$. Therefore the area of $\triangle ABC$ is $\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}$.

Difficulty: Medium-hard

NCTM Standard: Geometry Standard: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Right Triangle

At Jefferson Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer?

(A) 30% **(B)** 40% **(C)** 49% **(D)** 51% **(E)** 70%

2009 AMC 10 A, Problem #18—
"Of the 60 soccer players, 40% or $60 \times \frac{40}{100} = 24$ are also swimmers."

Solution

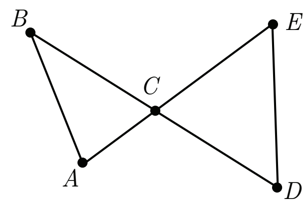
Answer (D): For every 100 children, 60 are soccer players and 40 are non-soccer players. Of the 60 soccer players, 40% or $60 \times \frac{40}{100} = 24$ are also swimmers, so 36 are non-swimmers. Of the 100 children, 30 are swimmers and 70 are non-swimmers. The fraction of non-swimmers who play soccer is $\frac{36}{70} \approx .51$, or 51%.

Difficulty: Medium-hard

NCTM Standard: Algebra Standard: use mathematical models to represent and understand quantitative relationships.

Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Percent

Segment BD and AE intersect at C, as shown, $AB = BC = CD = CE$, and $\angle A = \frac{5}{2}\angle B$. What is the degree measure of $\angle D$?



(A) 52.5 **(B)** 55 **(C)** 57.5 **(D)** 60 **(E)** 62.5

2009 AMC 10 B, Problem #9—
"Notice that $\triangle ABC$ and $\triangle CDE$ are isosceles."

Solution

Answer (A): Because $\triangle ABC$ is isosceles, $\angle A = \angle C$. Because $\angle A = \frac{5}{2}\angle B$, we have $\frac{5}{2}\angle B + \frac{5}{2}\angle B + \angle B = 180°$, so $\angle B = 30°$. Therefore $\angle ACB = \angle DCE = 75°$. Because $\triangle CDE$ is isosceles, $2\angle D + 75° = 180°$, so $\angle D = 52.5°$.

Difficulty: Medium

NCTM Standard: Geometry Standard: analyze properties and determine attributes of two- and three-dimensional objects.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Isosceles Triangle

Points A and C lie on a circle centered at O, each of $\overline{BA}$ and $\overline{BC}$ are tangent to the circle, and $\triangle ABC$ is equilateral. The circle intersects $\overline{BO}$ at D. What is $\frac{BD}{BO}$?

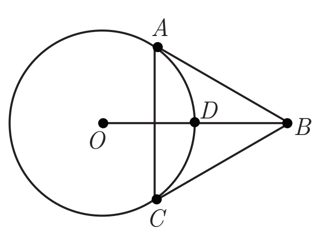
(A) $\frac{\sqrt{2}}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{3}}{3}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

2009 AMC 10 B, Problem #16— "$\triangle BCO$ is a right triangle with a 30° angle at B."

Solution

Answer (B): Let the radius of the circle be r. Because $\triangle BCO$ is a right triangle with a 30° angle at B, the hypotenuse $\overline{BO}$ is twice as long as $\overline{OC}$, so $BO = 2r$. It follows that $BD = 2r - r = r$, and

$$\frac{BD}{BO} = \frac{r}{2r} = \frac{1}{2}.$$

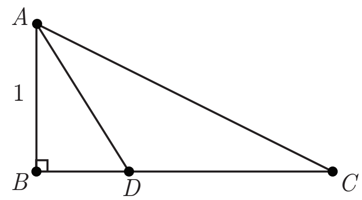


Difficulty: Medium-hard

NCTM Standard: Geometry Standard: analyze properties and determine attributes of two- and three-dimensional objects.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > 30-60-90 Triangle

Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?



(A) $\frac{\sqrt{3}-1}{2}$ **(B)** $\frac{\sqrt{5}-1}{2}$ **(C)** $\frac{\sqrt{5}+1}{2}$ **(D)** $\frac{\sqrt{6}+\sqrt{2}}{2}$ **(E)** $2\sqrt{3}-1$

2009 AMC 10 B, Problem #20— "Angle Bisector Theorem."

Solution

Answer (B): By the Pythagorean Theorem, $AC = \sqrt{5}$. By the Angle Bisector Theorem, $\frac{BD}{AB} = \frac{CD}{AC}$. Therefore $CD = \sqrt{5} \cdot BD$ and $BD + CD = 2$, from which

$$BD = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

OR

Let $\overline{DE}$ be an altitude of $\triangle ADC$. Then note that $\triangle ABD$ is congruent to $\triangle AED$, and so $AE = 1$. As in the first solution $AC = \sqrt{5}$. Let $x = BD$. Then $DE = x$, $EC = \sqrt{5} - 1$, and $DC = 2 - x$. Applying the Pythagorean Theorem to $\triangle DEC$ yields $x^2 + \left(\sqrt{5}-1\right)^2 = (2-x)^2$, from which $x = \frac{\sqrt{5}-1}{2}$.

Difficulty: Medium-hard

NCTM Standard: Geometry Standard: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Trigonometry > Angles > Angle Bisector Theorem

AMC 12 Student Practice Questions

What number is one third of the way from $\frac{1}{4}$ to $\frac{3}{4}$?

(A) $\frac{1}{3}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{2}{3}$

2009 AMC 12 A, Problem #3—
"Find one third of the difference between $\frac{1}{4}$ and $\frac{3}{4}$, then calculate the required number."

Solution

Answer (B): The number is

$$\frac{1}{4}+\frac{1}{3}\left(\frac{3}{4}-\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}.$$

Difficulty: Medium-easy
NCTM Standard: Number and Operations Standard: compare and contrast the properties of numbers and number systems, including the rational and real numbers, and understand complex numbers as solutions to quadratic equations that do not have real solutions.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Fraction

Functions f and g are quadratic, $g(x) = -f(100-x)$, and the graph of g contains the vertex of the graph of f. The four x-intercepts on the two graphs have x-coordinates x_1, x_2, x_3, and x_4, in increasing order, and $x_3-x_2 = 150$. The value of x_4-x_1 is $m+n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. What is $m+n+p$?

(A) 602 **(B)** 652 **(C)** 702 **(D)** 752 **(E)** 802

2009 AMC 12 A, Problem #23—
"Write $f(x)$ in "completing-the-square" form, then express $g(x)$ in terms of $f(x)$."

Solution

Answer (D): Let (h, k) be the vertex of the graph of f. Because the graph of f intersects the x-axis twice, we can assume that $f(x) = a(x-h)^2+k$ with $\frac{-k}{a} > 0$. Let $s = \sqrt{\frac{-k}{a}}$; then the x-intercepts of the graph of f are $h \pm s$. Because $g(x) = -f(100-x) = -a(100-x-h)^2 - k$, it follows that the x-intercepts of the graph of g are $100-h \pm s$.
The graph of g contains the point (h, k); thus

$$k = f(h) = g(h) = -a(100-2h)^2 - k,$$

from which $h = 50 \pm \frac{\sqrt{2}}{2}s$. Regardless of the sign in the expression for h, the four x-intercepts in order are

$$50-s\left(1+\frac{\sqrt{2}}{2}\right) < 50-s\left(1-\frac{\sqrt{2}}{2}\right) < 50+s\left(1-\frac{\sqrt{2}}{2}\right) < 50+s\left(1+\frac{\sqrt{2}}{2}\right).$$

Because $x_3 - x_2 = 150$, it follows that $150 = s(2-\sqrt{2})$, that is $s = 150\left(1+\frac{\sqrt{2}}{2}\right)$. Therefore $x_4 - x_1 = s(2+\sqrt{2}) = 450 + 300\sqrt{2}$, and then $m+n+p = 450+300+2 = 752$.

Difficulty: Hard
NCTM Standard: Algebra Standard: understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions.
Mathworld.com Classification: Algebra > Polynomials > Quadratic Polynomial

Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

(A) 89 **(B)** 90 **(C)** 120 **(D)** 2^{10} **(E)** $2^2 3^8$

2009 AMC 12 B, Problem #21— "Try cases for 1 woman, 2 women, 3 women first."

Solution

Answer (A): Let S_n denote the number of ways that n women in n seats can be reseated so that each woman reseats herself in the seat she occupied before or a seat next to it. It is easy to see that $S_1 = 1$ and $S_2 = 2$. Now consider the case with $n \geq 3$ women, and focus on the woman at the right end of the line. If this woman sits again in this end seat, then the remaining $n-1$ women can reseat themselves in S_{n-1} ways. If this end woman sits in the seat next to hers, then the former occupant of this new seat must sit on the end. Then the remaining $n-2$ women can seat themselves in S_{n-2} ways. Thus for $n \geq 3$, $S_n = S_{n-1} + S_{n-2}$. Therefore $(S_1, S_2, \ldots, S_{10}) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89)$, which are some of the first few terms of the Fibonacci Sequence. Thus $S_{10} = 89$.

Difficulty: Hard
NCTM Standard: Number and Operations Standard: develop an understanding of permutations and combinations as counting techniques.
Mathworld.com Classification: **Discrete Mathematics > Recurrence Equations > Fibonacci Number**

For how many values of x in $[0, \pi]$ is $\sin^{-1}(\sin 6x) = \cos^{-1}(\cos x)$?
Note: The functions $\sin^{-1} = \arcsin$ and $\cos^{-1} = \arccos$ denote inverse trigonometric functions.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

2009 AMC 12 B, Problem #24—
"If $0 \le x \le \pi/12$, then $\sin^{-1}(\sin 6x) = 6x$."

Solution

Answer (B): Let $f(x) = \sin^{-1}(\sin 6x)$ and $g(x) = \cos^{-1}(\cos x)$. If $0 \le x \le \pi$, then $g(x) = x$. If $0 \le x \le \pi/12$, then $f(x) = 6x$. Note also that $\sin\left(6\left(\frac{\pi}{6} - x\right)\right) = \sin 6x$, $\sin\left(6\left(\frac{\pi}{3} - x\right)\right) = -\sin 6x$, and $\sin\left(6\left(\frac{\pi}{3} + x\right)\right) = \sin 6x$, from which it follows that $f(\frac{\pi}{6} - x) = f(x)$, $f(\frac{\pi}{3} - x) = -f(x)$, and $f(\frac{\pi}{3} + x) = f(x)$. Thus the graph of $y = f(x)$ has period $\frac{\pi}{3}$ and consists of line segments with slopes of 6 or -6 and endpoints at $((4k+1)\frac{\pi}{12}, \frac{\pi}{2})$ and $((4k+3)\frac{\pi}{12}, -\frac{\pi}{2})$ for integer values of k. The graphs of f and g intersect twice in the interval $[0, \frac{\pi}{6}]$ and twice more in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$. If $\frac{\pi}{2} < x \le \pi$, then $g(x) = x > \frac{\pi}{2}$, so the graphs of f and g do not intersect.

Difficulty: Hard
NCTM Standard: Algebra Standard: understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions.
Mathworld.com Classification: Calculus and Analysis > Special Functions > Trigonometric Functions > Trigonometric Functions